TESTABLE IMPLICATIONS OF TRANSLATION INVARIANCE AND HOMOTHETICITY: VARIATIONAL, MAXMIN, CARA AND CRRA PREFERENCES

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Abstract. We provide revealed preference axioms that characterize models of translation invariant preferences. In particular, we characterize the models of variational, maxmin, CARA and CRRA utilities. In each case we present a revealed preference axiom that is satisfied by a dataset if and only if the dataset is consistent with the corresponding utility representation. Our results complement traditional exercises in decision theory that take preferences as primitive.

1. Introduction

We work out the testable implications of models with translation invariant preferences. Given a finite dataset on purchases of state-contingent assets, we give a revealed preference axiom that describes the datasets that are consistent with different models of translation invariant preferences.

These models include risk neutral variational preferences (Maccheroni et al., 2006), risk neutral maxmin preferences (Gilboa and Schmeidler, 1989), and subjective expected utility preferences with constant absolute risk aversion: so-called CARA preferences. Analogously to the CARA case, we also work out the testable implications of subjective expected utility preferences with constant relative risk aversion: so-called CRRA preferences (these form the "homothetic" class alluded to in the title). The models have been used by economists for different purposes. Variational and maxmin preferences are the most commonly-used models of ambiguity aversion. They are also used

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to capture model robustness (Hansen and Sargent, 2008). CARA and CRRA preferences are very common in applied work in macroeconomics and finance, among other fields.

Our contribution is to start from finite data on state-contingent consumption purchases, such as one would observe from a market experiment on choice under uncertainty (Hey and Pace, 2014; Ahn et al., 2014; Bayer et al., 2012). We describe the datasets that are rationalizable as consistent with a preference relation that satisfies translation invariance. When we say that we describe the datasets that are rationalizable, we mean that we provide a property, a "revealed preference axiom," that the data satisfy if and only if they are consistent with the theory in question.

The models we study have well known axiomatizations when one takes preferences as primitive, but not when one takes consumption data as given. The axiomatization of variational preferences is due to Maccheroni et al. (2006) (see also Grant and Polak (2013) and Siniscalchi (2009) for variations on their arguments). The axiomatization of maxmin is due to Gilboa and Schmeidler (1989). Our focus is on behavior in the market, not on preferences. The primitive is a finite list of purchases of state-contingent payments, each one made at a different price vector.

In contrast with most papers on ambiguity, we do not work in the Anscombe-Aumann framework. For this reason, we must restrict attention to risk-neutral variational and maxmin preferences. The Anscombe-Aumann framework would allow us to identify the utility function over outcomes. Without this recourse, we are restricted to the risk-neutral case. If we were to add the "observation" of a utility function to our datasets, then we could proceed as in the Anscombe-Aumann approach.

It would of course be desirable to obtain results without the assumption of risk neutrality; but these are likely difficult to come by. One exception is the case of maxmin utility with two states: we give a characterization of the data sets that are rationalizable with risk-averse (concave utility over money) maxmin in Section 6. The two-state case is restrictive, but probably of interest for experiments on ambiguity: some of the most basic experiments illustrating ambiguity aversion involve two states.

As is typical in revealed preference models, concavity often adds no empirical content to the model in question. In the case of subjective uncertainty, with our hypothesis of risk-neutrality, this implies that ambiguity aversion in the sense of Gilboa and Schmeidler (1989) adds no empirical content. Of course, this is due to the linear structure of prices and would break down in a more general framework.

The closest papers to ours are Varian (1988), Bayer et al. (2012) and Polisson et al. (2013). Our results on CARA and CRRA are close to Varian (1988). The main difference is that Varian considers the case of objective probabilities, not subjective. Bayer et al. (2012) and Polisson et al. (2013) look at the testable implications of models of ambiguity aversion for the same kinds of data that we assume in this paper. They give a characterization in terms of the solution of a system of inequalities. Our contribution is different because we give a revealed preference axiom that has to be satisfied for the data to be rationalizable. It can be written in the UNCAF form, the importance of which is developed by (Chambers et al., 2014).

The papers by (Kubler et al., 2014) and (Echenique and Saito, 2015) are also related. Kubler et al solve the same problem as we do here, but for the case of expected utility theory with known (objective) probabilities over states. Echenique and Saito solve the problem for subjective expected utility.

2. Definitions.

Let S be a finite set of states of the world. An *act* is a function from S into $\mathbf{R}$. So $\mathbf{R}^S$ is the set of acts. An act can be interpreted as a state-contingent monetary payment. Define $\|x\|_1 = \sum_s x_s$. $\Delta(S)$ represents the set of probability distributions on S, *i.e.* $\Delta(S) = \{\pi \in \mathbf{R}^S_+ : \sum_s \pi_s = 1\}$.

A *preference relation* on $\mathbf{R}^S$ is a binary relation $\succeq$ that is complete and transitive. Given a preference relation $\succeq$, we denote by $\succ$ the strict part of $\succeq$. A function $u : \mathbf{R}^S \to \mathbf{R}$ defines a preference relation $\succeq$ by $x \succeq y$ if and only if $u(x) \geq u(y)$. We say that u *represents* $\succeq$, or that it is a utility function for $\succeq$.

A preference relation $\succeq$ on $\mathbf{R}^S$ is *locally nonsatiated* if for every x and every $\varepsilon > 0$ there is y such that $\|x - y\| < \varepsilon$ and $y \succ x$.

3. Preferences, utilities, and data.

A *data set* D is a finite collection $\{(p^k, x^k)\}_{k=1}^K$, where each $p^k \in \mathbf{R}^S_{++}$ is a vector of strictly positive (Arrow-Debreu) prices, and each $x^k \in \mathbf{R}^S$ is an act. The interpretation of a dataset is that each pair (p^k, x^k) consists of an act x^k chosen from the budget $\{x \in \mathbf{R}^S : p^k \cdot x \leq p^k \cdot x^k\}$ of affordable acts.[1]

A data set $\{(p^k, x^k)\}_{k=1}^K$ is *rationalizable* by a preference relation $\succeq$ if $x^k \succeq x$ whenever $p^k \cdot x^k \geq p^k \cdot x$. So a data set is rationalizable by a preference relation when the choices in the dataset would have been optimal for that preference relation.

A data set $\{(p^k, x^k)\}_{k=1}^K$ is *rationalizable* by a utility function u if it is rationalizable by the preference relation represented by u. So a data set is rationalizable by a utility function when the choices in the dataset would have maximized that utility function in the relevant budget set.

A preference relation $\succeq$ is *translation invariant* if for all $x, y \in \mathbf{R}^S$ and all $c \in \mathbf{R}$, we have $x \succeq y$ if and only if $x + (c, \ldots, c) \succeq y + (c, \ldots, c)$.

A preference relation $\succeq$ is *homothetic* if for all $x, y \in \mathbf{R}^S$ and all $\alpha > 0$, we have $x \succeq y$ if and only if $\alpha x \succeq \alpha y$.

A preference relation $\succeq$ is a *risk-neutral variational preference* if there is a convex and lower semicontinuous function $c : \Delta(S) \to \mathbf{R} \cup \{+\infty\}$ for which there is $\pi \in \Delta(S)$ satisfying $c(\pi) < +\infty$, $c(\pi) < +\infty$ implies for all $s \in S$, $\pi_s > 0$, such that the utility function

$$\inf_{\pi \in \Delta(S)} \pi \cdot x + c(\pi)$$

represents $\succeq$. If a data set is rationalizable by a variational preference relation, we will say that the dataset set is *risk-neutral variational-rationalizable.*

A special case of variational preference is maxmin: A preference relation is *risk-neutral maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$ for which for each $\pi \in \Pi$ and all $s \in S$, $\pi_s > 0$, such that the utility function

$$\inf_{\pi \in \Pi} \pi \cdot x$$

[1]Arrow-Debreu prices make sense in a setting of complete markets and absence of arbitrage. Arrow-Debreu prices can then be recovered from asset prices. We also imagine experimental data from markets in which Arrow-Debreu securities are traded (Hey and Pace, 2014; Ahn et al., 2014; Bayer et al., 2012).

represents $\succeq$. If a data set is rationalizable by a risk neutral maxmin preference relation, we will say that the dataset set is *(risk-neutral) maxmin-rationalizable.*

A utility $u : \mathbf{R}^S \to \mathbf{R}$ is *constant absolute risk aversion (CARA)* if there is $a > 0$ and $\pi \in \Delta(S)$ for which for all $s \in S$, $\pi_s > 0$, and

$$u(x) = \sum_{s \in S} \pi_s \left(-\exp(-ax)\right).$$

Note that CARA is a special case of subjective expected utility.

A utility $u : \mathbf{R}^S \to \mathbf{R}$ is *constant relative risk aversion (CRRA)* if there is $a \in (0, 1)$ and $\pi \in \Delta(S)$ for which for all $s \in S$, $\pi_s > 0$, and

$$u(x) = \sum_{s \in S} \pi_s \left(\frac{x^{1-a}}{1-a}\right).$$

If a data set is rationalizable by a CARA (CRRA) utility, we will say that the dataset set is *CARA (CRRA) rationalizable.*

4. Variational preferences

We present the results on variational and maxmin rationalizability as Theorems 1 and 4. In each case, the model in question assumes a linear utility index: so the model captures ambiguity aversion but risk neutrality. These results beg the question of the empirical content of risk aversion together with ambiguity aversion. In Section 6 we present a result on maxmin utility with risk aversion. It is restricted to environments with two states.

1. **Theorem.** *The following statements are equivalent:*

(1) *Dataset D is rationalizable by a locally nonsatiated, translation invariant preference.*
(2) *Dataset D is rationalizable by a continuous, strictly increasing, concave utility function satisfying the property $u(x + (c, \dots, c)) = u(x) + c$.*
(3) *Dataset D is risk-neutral variational-rationalizable.*
(4) *For every $l = 1, \dots, M$, and every sequence $\{k_l\} \subseteq \{1, \dots, K\}$,*

$$\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) \geq 0,$$

where addition is modulo M, as usual.

Note that the equivalence between (2) and (3) is due to Maccheroni et al. (2006).

The equivalence of (1) and (2) implies that if data are rationalizable by a translation invariant preference, they are also rationalizable by a risk-neutral variational preference which is ambiguity averse, in the sense of Gilboa and Schmeidler (1989). Thus, owing to linear pricing, ambiguity aversion adds no empirical content to the variational model.

2. *Remark.* The preceding result can be generalized. Suppose we were interested in the testable implications of preferences which are β-translation invariant, for some $\beta \geq 0$, $\beta \neq 0$. That is, we want to know whether for all x, y, we have $x \succeq y$ if and only if for all t, $x + t\beta \succeq y + t\beta$. Define the seminorm $\|x\|_1^\beta = \sum_i |\beta_i x_i|$. Then it is an easy exercise to verify that the testable implications of β-translation invariance are given by equation (4), replacing $\|\cdot\|_1$ with $\|\cdot\|_1^\beta$.

3. *Remark.* The test in (4) is related to cyclic monotonicity. This is similar to the test given by Brown and Calsamiglia (2007) for quasilinear preferences (and to a result in (Rockafellar, 1997) characterizing superdifferentials of concave functions).

We now turn our attention to maxmin preferences.

We say that a function $u : \mathbf{R}^S \to \mathbf{R}$ is linearly homogeneous if for all $x \in \mathbf{R}^S$ and all $\alpha > 0$, we have $u(\alpha x) = \alpha u(x)$.

4. **Theorem.** *The following statements are equivalent:*

(1) *Dataset D is rationalizable by a locally nonsatiated, homothetic and translation invariant preference.*
(2) *Dataset D is rationalizable by a continuous, strictly increasing, linearly homogeneous and concave utility function satisfying the property that $u(x + (c, \ldots, c)) = u(x) + c$.*
(3) *Dataset D is risk-neutral maxmin-rationalizable.*
(4) *For every k and l,*

$$\frac{p^k}{\|p^k\|_1} \cdot x^k \leq \frac{p^l}{\|p^l\|_1} \cdot x^k.$$

The equivalence between (2) and (3) is due to Gilboa and Schmeidler (1989). Here we prove it through an application of Theorem 1.

5. CARA and CRRA

The previous section considers translation invariance and homotheticity as general properties of preferences in choice under uncertainty. Here we focus on the case of subjective expected utility. So we consider models in which the agent has a single prior over states, and maximizes expected utility. The prior is unknown though, and must be inferred from her choices. In the subjective expected utility case, translation invariance gives rise to CARA preferences, and homotheticity to CRRA.

5. **Theorem.** *A dataset D is CARA rationalizable if and only if there is $\alpha^* > 0$ such that* (1) *holds; and CRRA rationalizable if and only if there is $\alpha^* \in (0,1)$ such that* (2) *holds.*

$$\alpha^*(x_t^k - x_s^k + x_s^{k'} - x_t^{k'}) = \log\left(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}\right) \tag{1}$$

$$\alpha^* \log\left(\frac{x_t^k}{x_s^k}\frac{x_s^{k'}}{x_t^{k'}}\right) = \log\left(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}\right) \tag{2}$$

The conditions in Theorem 5 may look like existential conditions: essentially Afriat inequalities. Afriat inequalities are indeed the source of equations (1) and (2), as evidenced by the proof of Theorem 5, but note that the statements are equivalent to non-existential statements: Equation (1) says that when $(x_t^k - x_s^k + x_s^{k'} - x_t^{k'}) \neq 0$,

$$\frac{\log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}})}{(x_t^k - x_s^k + x_s^{k'} - x_t^{k'})}$$

is independent of k, t, k' and s; and that when $(x_t^k - x_s^k + x_s^{k'} - x_t^{k'}) = 0$ then $\log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}) = 0$. Similarly for equation (2).

It is worth pointing out that, except in the case when for all observations, all prices are equal, and consumption of all goods are equal, equation (1) can have only one solution. Hence, risk preferences are uniquely identified.

The next corollary also shows that beliefs are identified. Recall that a CARA utility is defined by a pair (a, π), with $a > 0$ and $\pi \in \Delta(S)$.

6. **Corollary.** *If (a, π) and (a', π') define CARA utilities that rationalize D, then $(a, \pi) = (a', \pi)$. Furthermore, $a = a'$ coincide with the unique solution to* (1). *Similarly for CRRA rationalizability and* (2).

6. Risk averse maxmin with two states

Theorem 4 is about risk neutral maxmin. Here we turn to maxmin with risk aversion. A preference relation is *maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$, where for each $\pi \in \Pi$ and each $s \in S$, $\pi_s > 0$, and a concave utility $u : \mathbf{R}^S \to \mathbf{R}$ such that the utility function

$$\inf_{\pi \in \Pi} \sum_{s=1,2} \pi_s u(x_s)$$

represents $\succeq$. If a data set is rationalizable by a maxmin preference relation, we will say that the dataset set is *maxmin-rationalizable*.

Let K_0 be the set of all k such that $x_1^k = x_2^k$. Let K_1 be the set of all k such that $x_1^k < x_2^k$, and K_2 be the set of all k such that $x_1^k > x_2^k$. Note that $K = K_0 \cup K_1 \cup K_2$.

Say that a sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ is *balanced* if each k appears as k_i (on the left of the pair) the same number of times it appears as k_i' (on the right).

Given a sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, consider the following notation: Let $I_{l,s} = \{i : k_i \in K_l \text{ and } s_i = s\}$, $I'_{l,s} = \{i : k_i' \in K_l' \text{ and } s_i' = s\}$, for $l = 0, 1, 2$ and $s = 1, 2$.

Strong Axiom of Revealed Maxmin Expected Utility (SARMEU): *For any balanced sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

(1) $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ *for all i;*
(2) $|I_{0,1}| + |I_{1,1}| - |I'_{1,1}| = |I'_{0,1}| + |I'_{2,1}| - |I_{2,1}| \leq 0$

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

7. **Theorem.** *A dataset is maxmin rationalizable if and only if it satisfies SARMEU.*

6.1. **Discussion.** Echenique and Saito (2015) show that the following axiom characterizes rationalizability by subjective expected utility.

Strong Axiom of Revealed Subjective Expected Utility (SARSEU): *For any balanced sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

(1) $x^{k_i}_{s_i} > x^{k'_i}_{s'_i}$ *for all* i*;*
(2) $|I_{0,1}| + |I_{1,1}| + |I_{2,1}| = |I'_{0,1}| + |I'_{1,1}| + |I'_{2,1}|$

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \leq 1.$$

Since $S = \{1, 2\}$, condition (2) of SARSEU is equivalent to the condition that $s = 1$ appears as s_i (on the left of the pair) the same number of times it appears as s'_i (on the right). That is, $|I_{0,1}|+|I_{1,1}|+|I_{2,1}|$ counts all i with $s_i = 1$ and $|I'_{0,1}| + |I'_{1,1}| + |I'_{2,1}|$ counts all i with $s'_i = 1$. When these quantities are equal we obtain that the number of times $s = 2$ appears as s_i also equals the number of times it appears as s'_i. The reason is that $n-(|I_{0,1}|+|I_{1,1}|+|I_{2,1}|) = |I_{0,2}| + |I_{1,2}| + |I_{2,2}|$; and similarly for the sum of $I'_{l,s}$.[2]

Inspection of SARSEU and SARMEU yields the following

8. **Proposition.** *If a dataset satisfies SARSEU then it satisfies SARMEU.*

For a dataset to be maxmin rationalizable, but inconsistent with subjective expected utility, it needs to contain a sequence in the conditions of SARSEU in which $|I_{0,1}|+|I_{1,1}|+|I_{2,1}| = |I'_{0,1}|+|I'_{1,1}|+|I'_{2,1}|$, but where $|I_{0,1}|+|I_{1,1}|-|I'_{1,1}| > 0$.

As we have emphasized, the result in Theorem 7 is for two states. There are two simplifications afforded by the assumption of two states, and the two are crucial in obtaining the theorem. The first is that with two states there are only two extreme priors to any set of priors. With the assumption that u is monotonic, one can know which of the two extremes is relevant to evaluate any given act.[3] The second simplification is a bit harder to see, but it comes from the fact that one can normalize the probability of one state to be one and only keep track of the probability of the other state. Then the property of being an extreme prior carries over to the probability of the state that is left "free."[4]

[2]By the same reason, condition (2) of SARSEU is equivalent to the condition that $|I_{0,2}| + |I_{2,2}| + |I_{1,2}| = |I'_{0,2}| + |I'_{2,2}| + |I'_{1,2}|$.
[3]This would also be true in the model of Schmeidler (1989), whose ambiguity averse counterpart is equivalent to MEU in the case of two states.
[4]This can be seen in the proof of Lemma 9 when we go from $\bar{\pi} \geq \underline{\pi}$ to $\bar{\mu}_1 \geq \underline{\mu}_1$.

7. Proofs

7.1. **Proof of Theorem 1.** That (3) $\Longrightarrow$(1) is obvious. We shall first prove that (1) $\Longrightarrow$(4)

Suppose, towards a contradiction, D is a dataset satisfying (1) but not (4). Then we have a cycle $\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) < 0$. Let us without loss assume the sequence is $x^1, \dots, x^M$ so as to avoid cumbersome notation. Let $Z = \sum_{l=1}^{M} \frac{p^l}{\|p^l\|_1} \cdot (x^{l+1} - x^l) < 0$.

Define a new sequence $(y^1, \dots, y^M)$ inductively. Let $y^1 = x^1$, and let $y^k = x^k + (c^k, \dots, c^k)$ where c^k is chosen so that $\frac{p^k}{\|p^k\|_1} \cdot (y^{k+1} - y^k) = \frac{Z}{M}$. Specifically, $c^1 = 0$ and

$$c^{k+1} = c^k + \frac{Z}{M} - \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k)$$

for $k = 1, \dots, M-1$. Let $q^k = \frac{p^k}{\|p^k\|_1}$ and consider the dataset (q^k, y^k), $k = 1, \dots M$.

The original dataset is rationalizable by some locally non-satiated and translation invariant preference $\succeq$. It is easy to see that the same preference rationalizes the dataset (q^k, y^k). Indeed, if $q^k \cdot y^k \geq q^k \cdot y$ then $p^k \cdot x^k \geq p^k \cdot (y - (c^k, \dots, c^k))$, by definition of y^k and q^k. So $x^k \succeq (y - (c^k, \dots, c^k))$, and thus $y^k \succeq y$ by translation invariance of $\succeq$.

Observe that

$$\begin{aligned}
& \sum_{k=1}^{M-1} q^k \cdot (y^{k+1} - y^k) + q^M \cdot (y^1 - y^M) \\
= \; & \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) + \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot ((c^{k+1}, \dots, c^{k+1}) - (c^k, \dots, c^k)) \\
= \; & \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) + \sum_{k=1}^{M} \frac{(\sum_{s \in S} p^k_s)(c^{k+1} - c^k)}{\|p^k\|_1} \\
= \; & \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) \quad (\because \|p^k\|_1 = \sum_{s \in S} p^k_s) \\
= \; & Z \quad (\because \text{Definition of } Z),
\end{aligned}$$

and that $q^k \cdot (y^{k+1} - y^k) = Z/M$ for $k = 1, \dots, M-1$. Therefore, $q^M \cdot (y^1 - y^M) = Z/M$. In particular, $q^k \cdot (y^{k+1} - y^k) = Z/M < 0$ for $k = 1, \dots, M$ (mod M).

Thus $y^k \succ y^{k+1}$ as (q^k, y^k) is rationalizable by $\succeq$ and $\succeq$ is locally nonsatiated. This contradicts the transitivity of $\succeq$.

Now we show that (4) $\Longrightarrow$(2). Let $x \in \mathbf{R}^S$. Let Σ_x be the set of all subsequences $\{k_l\}_{l=1}^M \subseteq \{1, \ldots, K\}$ for which $k_1 = 1$ and define $x^{k_{M+1}} = x$. By (4), if $\{k_l\}_{l=1}^M \in \Sigma_x$ has a cycle (meaning that $k_l = k_{l'}$ for $l, l' \in \{1, \ldots, M\}$ with $l \neq l'$), then there is a shorter sequence $\{k_j\}_{j=1}^{M'} \in \Sigma_x$ with

$$\sum_{j=1}^{M'} \frac{p^{k_j}}{\|p^{k_j}\|_1} \cdot (x^{k_{j+1}} - x^{k_j}) \leq \sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}).$$

Therefore, $u(x) = \inf\{\sum_{l=1}^M \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) : \{k_l\}_{l=1}^M \in \Sigma_x\}$ is well defined, as the infimum can be taken over a finite set.

That $u : \mathbf{R}^S \to \mathbf{R}$ defined in this fashion is concave, strictly increasing and continuous is immediate. To see that it rationalizes the data, suppose that $p^k \cdot x^l \leq p^k \cdot x^k$. Then $\frac{p^k}{\|p^k\|_1} \cdot x^l \leq \frac{p^k}{\|p^k\|_1} \cdot x^k$. It is clear then by definition that $u(x^l) \leq u(x^k) + \frac{p^k}{\|p^k\|_1} \cdot (x^l - x^k) \leq u(x^k)$.

Finally, to show that $u(x) + (c, \ldots, c)) = u(x) + c$, note that for any p^k, we have $\frac{p^k}{\|p^k\|_1} \cdot (x + (c, \ldots, c)) = c + \frac{p^k}{\|p^k\|_1} \cdot x$. The result then follows by construction.

We end the proof by showing that (2) $\Longrightarrow$(3) Let $u : \mathbf{R}^S \to \mathbf{R}$ be as in the statement of (2). Define the concave conjugate of u by

$$\begin{aligned} f(\pi) &= \inf\{\pi \cdot x - u(x) : x \in \mathbf{R}^S\} \\ &= \inf\{\pi \cdot x + c\pi \cdot \mathbf{1} - u(x) - c : x \in \mathbf{R}^S, c \in \mathbf{R}\} \\ &= \inf\{\pi \cdot x - c(1 - \pi \cdot \mathbf{1}) - u(x) : x \in \mathbf{R}^S, c \in \mathbf{R}\}, \end{aligned}$$

where the second equality uses that $u(x + (c, \ldots, c)) = u(x) + c$. Now note that $f(\pi) = -\infty$ if $(1 - \pi \cdot \mathbf{1}) \neq 0$. Note also that the monotonicity of u implies that $f(\pi) = -\infty$ if there is s such that $\pi_S < 0$. One can also show that there is $\pi \in \Delta(S)$ for which $f(\pi) \in \mathbf{R}$.[5] Finally, observe that by strict monotonicity,

[5]For example, take π to support $\{z \in \mathbf{R}^S : u(z) \geq 0\}$ at 0. We claim that $f(\pi) = 0$. Suppose by means of contradiction that there is $x \in \mathbf{R}^S$ for which $\pi \cdot x < u(x)$. Observe that π supports $\{z \in \mathbf{R}^S : u(z) \geq \pi \cdot x\}$ at the action y which returns $\pi \cdot x$ in each state. Observe that $u(z) > \pi \cdot x$ implies $\pi \cdot z > \pi \cdot x$, by continuity of u and definition of the supporting hyperplane; that is, $\{z \in \mathbf{R}^S : u(z) \geq \pi \cdot x\} \subseteq \{z \in \mathbf{R}^S : \pi \cdot z \geq \pi \cdot x\}$ implies $\{z \in \mathbf{R}^S : u(z) > \pi \cdot x\} \subseteq \{z \in \mathbf{R}^S : \pi \cdot z > \pi \cdot x\}$ as the latter sets are the interiors of the former. Therefore, if $u(x) > \pi \cdot x$, we conclude $\pi \cdot x > \pi \cdot x$, a contradiction.

if there is $s \in S$ for which $\pi_s = 0$, then $f(\pi) = -\infty$. Hence we can consider the domain of f to be a subset of $\Delta(S)$. Moreover, $f(\pi) < +\infty$ implies for all $s \in S$, $\pi_s > 0$.

Now since u is continuous, it is a standard application of the separating hyperplane theorem to establish that $u(x) = \inf_{\pi \in \Delta(S)} \pi \cdot x - f(\pi)$. Since u rationalizes the dataset, the dataset is variational rationalizable.

7.2. **Proof of Theorem 4.** It is obvious that (3) $\Longrightarrow$(2) and that (2) $\Longrightarrow$(1). Hence, to show the theorem, it suffices to show that (4) implies (3) and that (1) implies (4).

For a dataset D, let $\pi^k = \frac{p^k}{\|p^k\|_1}$. It is easy to see that (4) $\Longrightarrow$(3). Let Π be the convex hull of $\{\pi^k : k = 1, \ldots, K\}$. Then it is immediate that $u(x) = \min_{\pi \in \Pi} \pi \cdot x$ rationalizes D. Moreover, for each $\pi \in \Pi$ and all $s \in S$, $\pi_s > 0$ because $\pi^k_s > 0$ for all $s \in S$ and $k \in K$.

We prove that (1) $\Longrightarrow$(4). Suppose that D satisfies (1) but not (4). Then there are k and l for which $\pi^l \cdot x^k < \pi^k \cdot x^k$. Let $\succeq$ be a preference relation as stated in (1). By homotheticity of $\succeq$, for any scalar $\theta > 0$, $\succeq$ rationalizes the data $D' \equiv \{(x^j, \pi^j) : j = 1, \ldots, K\} \cup \{(\theta x^l, \pi^l)\}$. To see this, observe that if $\pi^l \cdot x \leq \pi^l \cdot \theta x^l$, then $\pi^l \cdot \theta^{-1} x \leq \pi^l x^l$, so that $x^l \succeq \theta^{-1} \cdot x$, and by homogeneity, $\theta x^l \succeq x$. Now, for $\theta > 0$ sufficiently small, $\pi^l \cdot x^k < \pi^k \cdot x^k$ implies that

$$x^k \cdot (\pi^l - \pi^k) + \theta x^l \cdot (\pi^k - \pi^l) < 0.$$

So either $x^k \cdot (\pi^l - \pi^k) < 0$ or $\theta x^l \cdot (\pi^k - \pi^l) < 0$. Then the dataset D' violates (4) in Theorem 1, contradicting the fact that it is rationalized by $\succeq$, which is assumed to be translation invariant.

7.3. **Proof of Theorem 5.** The idea in the proof is to solve the first-order conditions for the unknown terms. Consider first the case of CARA. Let $\pi \in \Delta(S)$ and $\alpha > 0$ rationalize D. Then we know that x^k maximizes $\sum_s \pi_s \left(-\exp(-\alpha x_s)\right)$ subject to $p^k \cdot x \leq p^k \cdot x^k$. By considering the Lagrangean and the first order conditions, we may conclude that for every $s, t \in S$ and every $k \in \{1, \ldots, K\}$, we have

$$\frac{\pi_s \exp(-\alpha x^k_s)}{p^k_s} = \frac{\pi_t \exp(-\alpha x^k_t)}{p^k_t}.$$

Conclude that $\frac{p_s^k \pi_t}{p_t^k \pi_s} = \exp(-\alpha(x_s^k - x_t^k))$. By taking logs, the system becomes:

$$\log(\pi_s) - \log(\pi_t) + \alpha(x_t^k - x_s^k) = \log(p_s^k) - \log(p_t^k). \tag{3}$$

In the case of CRRA, the existence of a rationalizing π and parameter α imply a first-order condition of the form

$$\log(\pi_s) - \log(\pi_t) + \alpha \log(x_t^k / x_s^k) = \log(p_s^k) - \log(p_t^k). \tag{4}$$

We can denote $\log(\pi_s)$ by z_s in equations (3) and (4). Thus we obtain that D is rationalizable if and only if there exist $z_s \in \mathbf{R}$ and $\alpha > 0$ such that the following equation is solved for all s, t, k with $s \neq t$:

$$z_s - z_t + \alpha(y_t^k - y_s^k) = \log(p_s^k) - \log(p_t^k),$$

where $y_t^k = x_t^k$ for CARA rationalizability, and $y_t^k = \log x_t^k$ for CRRA rationalizability.

Now the necessity of the axioms is obvious. Let $k \neq k'$, then

$$\alpha(y_t^k - y_s^k) - \log(p_s^k/p_t^k) = z_s - z_t = \alpha(y_t^{k'} - y_s^{k'}) - \log(p_s^{k'}/p_t^{k'})$$

for any s and t. Thus

$$\alpha(y_t^k - y_s^k - y_t^{k'} + y_s^{k'}) = \log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}).$$

So (1) is satisfied for the case of CARA rationalizability, and (2) is satisfied for the case of CRRA rationalizability.

To prove sufficiency, let

$$d^p(s,t,k) = \log(p_s^k/p_t^k)$$
$$d^x(s,t,k) = y_s^k - y_t^k.$$

Let α^* be such that for all k, k', s, s' and t,

$$\alpha^*(y_t^k - y_s^k - y_t^{k'} + y_s^{k'}) = \log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}).$$

Then in particular, for all k, k', s, s' and t,

$$d^p(s,t,k) + \alpha^* d^x(s,t,k) + d^p(t,s,k') + \alpha^* d^x(t,s,k') = 0. \tag{5}$$

Note also that

$$
\begin{aligned}
(6) \qquad & d^p(s,t,k) + d^p(t,s',k) + d^p(s',s,k) \\
& \qquad + \alpha^*(d^x(s,t,k) + d^x(t,s',k) + d^x(s',s,k)) = 0.
\end{aligned}
$$

Fix $s_0 \in S$ and let $z_{s_0} \in \mathbf{R}$ be arbitrary. For any $s \in S$, define z_s by

$$z_s = z_{s_0} + \alpha^* d^x(s_0, s, k) + d^p(s, s_0, k),$$

for some k. In fact, by equation (5), this definition is independent of k because $d^p(s,s_0,k) + \alpha^* d^x(s,s_0,k) = d^p(s,s_0,k') + \alpha^* d^x(s,s_0,k')$.

Given this definition, note that

$$
\begin{aligned}
z_s - z_t &= \alpha^*(d^x(s_0,s,k) - d^x(s_0,t,k)) + d^p(s,s_0,k) - d^p(t,s_0,k) \\
&= \alpha^*(d^x(s_0,s,k) - d^x(s_0,t,k)) + d^p(s,s_0,k) - d^p(t,s_0,k) \\
&+ d^p(s,t,k) + d^p(t,s_0,k) + d^p(s_0,s,k) \\
&+ \alpha^*(d^x(s,t,k) + d^x(t,s_0,k) + d^x(s_0,s,k)) \\
&= d^p(s,t,k) + \alpha^* d^x(s,t,k),
\end{aligned}
$$

where the second equality uses equation (6).

Hence, with the constructed $(z_t)_{t \in S}$ we have

$$z_s - z_t + \alpha^*(y^k_t - y^k_s) = \log(p^k_s / p^k_t),$$

for all s, t, and k. The first-order conditions for rationalizability are therefore satisfied.

7.4. **Proof of Theorem 7.**

9. **Lemma.** *A dataset D is maxmin rationalizable if and only if there are v^k_s, λ^k, $s = 1, 2$, $k = 1, \ldots, K$, and $\bar{\pi}, \underline{\pi} \geq 0$ with $\bar{\pi} \geq \underline{\pi} > 0$, such that: for all k such that $x^k_s \neq x^k_{s'}$,*

$$
\begin{aligned}
\pi v^k_1 &= \lambda^k p^k_1 \\
v^k_2 &= \lambda^k p^k_2,
\end{aligned}
$$

where $\pi = \bar{\pi}$ *when* $x_1^k < x_2^k$ *and* $\pi = \underline{\pi}$ *when* $x_1^k > x_2^k$*; for all* k *such that* $x_s^k = x_{s'}^k$

$$\overline{\pi} v_1^k \geq \lambda^k p_1^k$$
$$\underline{\pi} v_1^k \leq \lambda^k p_1^k$$
$$v_2^k = \lambda^k p_2^k.$$

The numbers also satisfy that $v_s^k \leq v_{s'}^{k'}$ *when* $x_s^k > x_{s'}^{k'}$.

Proof. To prove sufficiency, let v_s^k, λ^k, $s = 1,2$, $k = 1,\dots,K$, and $\bar{\pi}, \underline{\pi} \geq 0$ with $\bar{\pi} \geq \underline{\pi}$ be as in the statement of the lemma. Define $\bar{\mu}, \underline{\mu} \in \Delta(S)$ as follows. Let $\bar{\mu}_1 = \bar{\pi}/(1+\bar{\pi})$, $\bar{\mu}_2 = 1/(1+\bar{\pi})$, and $\underline{\mu}_1 = \underline{\pi}/(1+\underline{\pi})$, $\underline{\mu}_2 = 1/(1+\underline{\pi})$. Since $\bar{\pi} \geq \underline{\pi}$,

$$\bar{\mu}_1 \geq \underline{\mu}_1 \text{ and } \bar{\mu}_2 \leq \underline{\mu}_2.$$

Define $\theta^k = \lambda^k/(1+\bar{\pi})$ if $x_1^k < x_2^k$ and $\theta^k = \lambda^k/(1+\underline{\pi})$ if $x_1^k > x_2^k$. Then we have that $\mu_s v_s^k = \theta^k p_s^k$, with $\mu_s = \bar{\mu}_s$ when $x_1^k < x_2^k$; and $\mu_s = \underline{\mu}_s$ when $x_1^k > x_2^k$.

Now consider k such that $x_1^k = x_2^k$. By the assumption, there exists π^k such that $\overline{\pi} \geq \pi^k \geq \underline{\pi}$ such that $\pi v_1^k = \lambda^k p_1^k$. Let $\mu_1^k = \pi/(1+\pi)$, $\mu_2^k = 1/(1+\pi)$, and $\theta^k = \lambda^k/(1+\pi)$. Since $\overline{\pi} \geq \pi \geq \underline{\pi}$,

$$\overline{\mu}_1 \equiv \frac{\overline{\pi}}{1+\overline{\pi}} \geq \frac{\pi^k}{1+\pi^k} \geq \frac{\underline{\pi}}{1+\underline{\pi}} \equiv \underline{\mu}_1.$$

Hence, there exists $\alpha^k \in [0,1]$, $\mu_1^k = \alpha^k \overline{\mu}_1 + (1-\alpha^k)\underline{\mu}_1$. Then, $\mu_2^k = 1 - \mu_1^k = \alpha^k(1-\overline{\mu}_1) + (1-\alpha^k)(1-\underline{\mu}_1) = \alpha^k \overline{\mu}_2 + (1-\alpha^k)\underline{\mu}_2$.

Given the numbers v_s^k it is now routine to define a correspondence ρ such that if $x \leq x'$, $y \in \rho(x)$ and $y' \in \rho(x')$ then $y \geq y' > 0$, and with $\rho(x_s^k) \ni v_s^k$. This gives a concave and increasing function u with $\partial u(c) = \rho(x)$. So $\frac{\theta^k p_s^k}{\mu_s} \in \partial u(x_s^k)$ for all s and k such that $x_1^k \neq x_2^k$. Moreover, for all k such that $x_1^k = x_2^k$

$$(\theta^k p_1^k, \theta^k p_2^k) \in \text{co}\left\{\left(\overline{\mu}_1 \partial u(x_1^k), \overline{\mu}_2 \partial u(x_2^k)\right), \left(\underline{\mu}_1 \partial u(x_1^k), \underline{\mu}_2 \partial u(x_2^k)\right)\right\}.$$

Hence the first and second order conditions are satisfied for maxmin rationalization.

We omit the proof of necessity. □

We will define matrices A, B, E such that there exist numbers $\{v_s^k\}$, $\{\lambda^k\}$, $\bar{\pi}$, $\underline{\pi}$ satisfying the conditions in Lemma 9 if and only if there exists a solution x to the system of inequalities $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$.

Let A be a matrix with $2K+2+K+1$ columns. The first $2K$ columns are labeled with a different pair (k,s). The next 2 columns are labeled $\bar{\pi}$ and $\underline{\pi}$. The next K columns are labeled with a $k \in \{1,\ldots,K\}$. Finally the last column is labeled p.

For each $(k,2)$ with $k \in K_0$, A has a row with all zero entries with the following exception. It has a 1 in the column labeled (k,s), among the first group of $2K$ columns. It has a -1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$.

For each (k,s) with $k \in K_1$, A has a row with all zero entries with the following exception. It has a 1 in the column labeled (k,s), among the first group of $2K$ columns. It has a -1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$. Finally, it has a 1 in the column labeled $\bar{\pi}$ if and only if if $s=1$. For each (k,s) with $k \in K_2$, A has a row defined as above. The only difference is that when $s=1$ then it has a 1 in the column labeled $\underline{\pi}$ instead of having a 1 in the column labeled $\bar{\pi}$.

Matrix A looks as follows:

$$
\begin{array}{r}
\\
\\
{\scriptstyle (k,s)\in(K_0,2)} \\
{\scriptstyle (k,s)\in(K_1,1)} \\
{\scriptstyle (k,s)\in(K_2,1)} \\
\\
\end{array}
\begin{array}{c}
\begin{array}{ccccc|cc|ccccc|c}
{\scriptstyle (1,1)} & \cdots & {\scriptstyle (k,s)} & \cdots & {\scriptstyle (K,S)} & {\scriptstyle \bar{\pi}} & {\scriptstyle \underline{\pi}} & {\scriptstyle 1} & \cdots & {\scriptstyle k} & \cdots & {\scriptstyle K} & {\scriptstyle p}
\end{array} \\
\left[
\begin{array}{ccccc|cc|ccccc|c}
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & -\log p_s^k \\
0 & \cdots & 1 & \cdots & 0 & 1 & 0 & 0 & \cdots & -1 & \cdots & 0 & -\log p_s^k \\
0 & \cdots & 1 & \cdots & 0 & 0 & 1 & 0 & \cdots & -1 & \cdots & 0 & -\log p_s^k \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots
\end{array}
\right]
\end{array}
$$

Let B be a matrix with the same number of columns as A. The columns of B are labeled like those of A. For each $(k,1)$ with $k \in K_0$, B has two rows. In the first row, B has a row with all zero entries with the following exception. It has a 1 in the column labeled (k,s), among the first group of $2K$ columns. It has a 1 in the column labeled $\bar{\pi}$. It has a -1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$. In the second row, B has a row with all zero entries with the following exception. It has a -1 in the column labeled (k,s), among the first group of $2K$ columns. It has a -1 in the column labeled $\underline{\pi}$. It has a 1 in the column labeled k. In the column labeled p it has $\log(p_s^k)$.

This first part of matrix B looks as follows:

$$
\begin{array}{r}
\\ \\ {\scriptstyle (k,s)\in(K_0,1)} \\ {\scriptstyle (k,s)\in(K_0,1)} \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc|cc|ccccc|c}
{\scriptstyle (1,1)} & \cdots & {\scriptstyle (k,s)} & \cdots & {\scriptstyle (K,S)} & {\scriptstyle \bar{\pi}} & {\scriptstyle \underline{\pi}} & {\scriptstyle 1} & \cdots & {\scriptstyle k} & \cdots & {\scriptstyle K} & {\scriptstyle p}
\end{array}\\
\left[\begin{array}{ccccc|cc|ccccc|c}
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 1 & 0 & 0 & \cdots & -1 & \cdots & 0 & -\log p_s^k \\
0 & \cdots & -1 & \cdots & 0 & 0 & -1 & 0 & \cdots & 1 & \cdots & 0 & \log p_s^k \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots
\end{array}\right]
\end{array}
$$

In addition, B has a row for each pair $(x_s^k, x_{s'}^{k'})$ with $x_s^k > x_{s'}^{k'}$. The row for $x_s^k > x_{s'}^{k'}$ has all zeroes except for a 1 in column (k', s') and a -1 in column (k, s). Finally, B has one more row. This row as a 1 in the column for $\bar{\pi}$ and a -1 in the column for $\underline{\pi}$. This second part of matrix B looks as follows:

$$
\begin{array}{r}
\\ \\ {\scriptstyle x_s^k > x_{s'}^{k'}} \\ \\ {\scriptstyle \bar{\pi} \geq \underline{\pi}}
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|cc|ccccc|c}
{\scriptstyle (1,1)} & \cdots & {\scriptstyle (k,s)} & \cdots & {\scriptstyle (k',s')} & \cdots & {\scriptstyle (K,S)} & {\scriptstyle \bar{\pi}} & {\scriptstyle \underline{\pi}} & {\scriptstyle 1} & \cdots & {\scriptstyle k} & \cdots & {\scriptstyle K} & {\scriptstyle p}
\end{array}\\
\left[\begin{array}{ccccccc|cc|ccccc|c}
\vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & -1 & \cdots & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{array}\right]
\end{array}
$$

Let E be a matrix with the same number of columns as A, labeled as above, and a single row. The row has all zeroes except for a 1 in column p.

By Lemma 9, there is no rationalizing maxmin preference if and only if there is no solution to the system of inequalities $A \cdot x = 0$, $B \cdot x \geq 0$, and $E \cdot x > 0$.

Suppose that all $\log(p_s^k)$ are rational numbers. We shall use the following version of the Theorem of the Alternative, which can be found as Theorem 1.6.1 in (Stoer and Witzgall, 1970).

10. **Lemma.** *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong to a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.*

(1) *There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.*
(2) *There is $\eta \in \mathbf{F}^m$, $\theta \in \mathbf{F}^l$, and $\gamma \in \mathbf{F}^r$ such that $\eta \cdot A + \theta \cdot B + \gamma \cdot E = 0$; $\theta \geq 0$ and $\gamma > 0$.*

Then the non-existence of a solution to the system $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$ is equivalent to the existence of integer vectors η, θ and γ such that $\theta \geq 0$, $\gamma > 0$, and $\eta \cdot A + \theta \cdot B + \gamma E = 0$.

For a matrix D with $2K+2+K+1$ columns, let D_1 denote the submatrix corresponding to the first $2K$ columns, D_2 correspond to the next 2, D_3 to the next K, and D_4 to the last column. Note that, by construction of A, B and E, $\eta \cdot A + \theta \cdot B + \gamma E = 0$ implies that $\eta \cdot A_1 + \theta \cdot B_1 = 0$, $\eta \cdot A_2 + \theta \cdot B_2 = 0$, $\eta \cdot A_3 + \theta \cdot B_3 = 0$, $\eta \cdot A_4 + \theta \cdot B_4 + \gamma = 0$. In fact, we can without loss assume that η, θ and γ take values of -1, 0 or 1. (This assumption is without loss because we can replace each row of matrices A, B and E with as many copies as indicated by the corresponding vector η, θ or γ.)

From the existence of such vectors it follows that we can obtain a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$ with $x^{k_i}_{s_i} > x^{k'_i}_{s'_i}$. The source of each pair $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})$ is that the column (k_i, s_i) of A is multiplied by $\eta_{(k_i,s_i)} > 0$ and the column (k'_i, s'_i) of A is multiplied by $\eta_{(k'_i,s'_i)} < 0$. The vector η must then have $\eta_{(k_i,s_i)} > 0$ and $\eta_{(k'_i,s'_i)} > 0$, with a -1 in the first column and a 1 in the second.

We shall prove that the sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$ satisfies the properties stated in the axiom.

Firstly, $\eta \cdot A_3 + \theta \cdot B_3 = 0$ means that for each k, the number of is for which $k = k_i$ equals the number of is for which $k = k'_i$.

Secondly, $\eta \cdot A_2 + \theta \cdot B_2 = 0$ implies that:

$$\sum_{k \in K_1} \eta_{(k,1)} + \sum_{k \in K_0} \theta_{(k,1)} + \theta_{\overline{\pi} \geq \underline{\pi}} = 0$$
$$\sum_{k \in K_2} \eta_{(k,1)} - \sum_{k \in K_0} \theta'_{(k,1)} - \theta_{\overline{\pi} \geq \underline{\pi}} = 0,$$

where $\theta_{\overline{\pi} \geq \underline{\pi}}$ is the nonnegative weight on the row associated with $\overline{\pi} \geq \underline{\pi}$; $\theta_{(k,1)}$ and $\theta'_{(k,1)}$ are the nonnegative weights on the two rows associated with $(k,1)$ with $k \in K_0$. ($\theta_{(k,1)}$ and $\theta'_{(k,1)}$ corresponds to the first row and the second row, respectively). Note that

$$\begin{aligned}
\sum_{k \in K_1} \eta_{(k,1)} &= |\{i : k_i \in K_1, s = 1\}| - |\{i : k'_i \in K_1, s = 1\}| \equiv |I_{1,1}| - \left|I'_{1,1}\right|, \\
\sum_{k \in K_2} \eta_{(k,1)} &= |\{i : k_i \in K_2, s = 1\}| - |\{i : k'_i \in K_2, s = 1\}| \equiv |I_{2,1}| - \left|I'_{2,1}\right|, \\
\sum_{k \in K_0} \theta_{(k,1)} &= |\{i : k_i \in K_0, s = 1\}| \equiv |I_{0,1}|, \\
\sum_{k \in K_0} \theta'_{(k,1)} &= |\{i : k'_i \in K_0, s = 1\}| \equiv \left|I'_{0,1}\right|.
\end{aligned}$$

Hence,

$$|I_{1,1}| - |I'_{1,1}| + |I_{0,1}| = |I'_{2,1}| - |I_{2,1}| + |I'_{0,1}| \leq 0.$$

Therefore the sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$ satisfies the second property stated in the axiom. Finally, since $\eta \cdot A_4 + \theta \cdot B_4 + \gamma = 0$,

$$\begin{aligned}
0 &> -\gamma \\
&= \eta \cdot A_4 + \theta \cdot B_4 \\
&= \sum_{(k,s)\in(K_0,2)\cup(K_1\cup K_2,1)} \eta_{(k,s)}(-\log p^k_s) + \sum_{k\in K_0} \theta_{(k,1)}(-\log p^k_1) + \sum_{k\in K_0} \theta'_{(k,1)} \log p^k_1 \\
&= \sum_{i=1}^n \log \frac{p^{k'_i}_{s'_i}}{p^{k_i}_{s_i}}
\end{aligned}$$

Hence

$$\prod_{i=1}^n \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} > 1.$$

The above proof assumes that the log of prices is rational. The proof of the theorem follows along the same lines as Echenique and Saito (2015). Specifically, we have shown the following

11. **Lemma.** *If $\{(x^k, p^k)\}$ is a dataset satisfying SARMEU, in which $\log p^k \in \mathbf{Q}$ for all k, then the dataset is maxmin rationalizable.*

One can then prove the following

12. **Lemma.** *If $\{(x^k, p^k)\}$ is a dataset that satisfies SARMEU, and $\varepsilon > 0$ then there is a collection of prices $\{q^k\}$ such that $\log q^k \in \mathbf{Q}$, $\|p^k - q^k\| < \varepsilon$, and the dataset $\{(x^k, q^k)\}$ satisfies SARMEU.*

The proof of Lemma 12 is exactly as in (Echenique and Saito, 2015).

Lemma 11 establishes the result in datasets in which the log of prices is rational. Consider an arbitrary data set $\{(x^k, p^k)\}$, with prices that may not be rational.

Suppose towards a contradiction that the dataset satisfies SARMEU, but that it is not maxmin rational. Specifically then, by Lemma 9, suppose that there is no solution to the system $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$. Then by Lemma 10 there are real vectors η, θ and γ such that $\theta \geq 0$, $\gamma > 0$, and $\eta \cdot A + \theta \cdot B + \gamma E = 0$.

Let $\{q^k\}$ be vectors of prices such that the dataset $\{(x^k, q^k)\}$ satisfies SARMEU and $\log q^k_s \in \mathbf{Q}$ for all k and s. (Such $\{q^k\}$ exists by Lemma 12.) Furthermore, the prices q^k can be chosen arbitrarily close to p^k. Construct matrices A', B', and E' from this dataset in the same way as A, B, and E above. Note that only the prices are different in $\{(x^k, q^k)\}$ compared to $\{(x^k, p^k)\}$. So $E' = E$, $B'_i = B_i$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 and B'_4 may be different from A_4 and B_4, respectively.

By Lemma 12, we can choose prices q^k such that $|\eta \cdot A'_4 + \theta \cdot B'_4 - (\eta \cdot A_4 + \theta \cdot B_4)| < \gamma/2$. We have shown that $\eta \cdot A_4 + \theta \cdot B_4 = -\gamma$, so the choice of prices q^k guarantees that $\eta \cdot A'_4 + \theta \cdot B'_4 < 0$. Let $\gamma' = -\eta \cdot A'_4 - \theta \cdot B'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \gamma' E_i = 0$ for $i = 1, 2, 3$. Hence

$$\eta \cdot A'_4 + \theta \cdot B'_4 + \gamma' E_4 = \eta \cdot A'_4 + \theta \cdot B_4 + \gamma' = 0.$$

We also have that $\eta \geq 0$ and $\gamma' > 0$. Therefore θ, η, and γ' exhibit a solution to the dual system for dataset $\{(x^k, q^k)\}$, a contradiction with Lemma 11.

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